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THE PROBLEM IN TERMS OF A STRESS TENSOR FOR AN ANISOTROPIC MEDIUM†

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A new formulation of the problem of the mechanics of a strained solid in terms of a stress tensor, proposed in [1] and later developed in [2–5, etc.], is extended to anisotropic media, physically linear and non-linear, when using tensor bases connected with a certain anisotropy group [6] of mechanical properties. The case of transverse isotropy is considered in detail.

1. We will assume everywhere a rectangular Cartesian system of coordinates in R^3. For brevity will often write "tensor" instead of the "components of the tensor". To satisfy the conditions of compatibility for small strains ε_{ij} [7]

$$\eta_{ij} \equiv \epsilon_{ikl}\epsilon_{jmn}\,\varepsilon_{kn,lm}=0 \tag{1.1}$$

it is necessary and sufficient for the following conditions to be satisfied

$$\eta_{ij} \equiv \Delta\varepsilon_{ij}+\theta_{,ij}-\varepsilon_{ik,kj}-\varepsilon_{jk,ki}=0, \quad \theta\equiv\varepsilon_{ii} \tag{1.2}$$

It follows from (1.2) that

$$\begin{aligned}\eta^\circ &\equiv \eta_{ij}\delta_{ij} \equiv 2(\Delta\theta-\varepsilon_{mn,mn})=0\\ \eta_{ij,j} &\equiv (\Delta\theta-\varepsilon_{mn,mn})_{,i}=0\end{aligned} \tag{1.3}$$

Suppose a tensor basis is constructed for the group G, which characterizes a certain anisotropy of the mechanical properties. Each tensor that occurs in this basis is invariant under the group G [6]. We will use this basis to construct second-rank tensors $a_{ij}^{(\alpha)}$ ($\alpha=1,\ldots,N$; $N\leqslant 3$), which in sum comprise the unit tensor δ_{ij} and are pairwise orthogonal, i.e.

$$\sum_{\alpha=1}^{N} a_{ij}^{(\alpha)}=\delta_{ij}, \quad \frac{a_{ij}^{(\alpha)}a_{ij}^{(\beta)}}{a_{(\alpha)}a_{(\beta)}}=\delta_{\alpha\beta}, \quad a_{(\alpha)}\equiv\sqrt{a_{ij}^{(\alpha)}a_{ij}^{(\alpha)}} \tag{1.4}$$

Note also that the following relations hold

$$a_{ik}^{(\alpha)}a_{kj}^{(\beta)}=a_{ij}^{(\alpha)}\delta_{\alpha\beta} \tag{1.5}$$

We will now consider the incompatibility tensor η_{ij} (1.2). Its linear invariants are formed using the tensor basis of the group G

$$\eta_{(\alpha)}\equiv\eta_{ij}a_{ij}^{(\alpha)} \tag{1.6}$$

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It follows from the first relations of (1.3), (1.4) and (1.6) that

$$\eta^\circ \equiv \eta_{ij}\delta_{ij} \equiv \eta_{ij}\sum_{\alpha=1}^{N} a_{ij}^{(\alpha)} = \sum_{\alpha=1}^{N}\eta_{(\alpha)} \tag{1.7}$$

Suppose $\xi^{(\alpha)}$ $(\alpha = 1, \ldots, N)$ are certain, for the present, arbitrary numbers. Then the tensor H_{ij}, constructed as follows:

$$H_{ij} = \eta_{ij} + \sum_{\alpha=1}^{N}\xi^{(\alpha)}\eta_{(\alpha)}a_{ij}^{(\alpha)} \tag{1.8}$$

vanishes together with the tensor η_{ij}.

In fact, it follows from (1.2), according to (1.6), that

$$\eta_{(\alpha)} = 0 \tag{1.9}$$

It therefore follows from (1.8) that

$$H_{ij} = 0 \tag{1.10}$$

If (1.10) is satisfied, convoluting the tensor H_{ij} with the tensor of the basis $a_{ij}^{(\beta)}$ $(\beta = 1, \ldots, N)$ we obtain from (1.8) and the second relation of (1.4)

$$H_{ij}a_{ij}^{(\beta)} = \eta_{(\beta)} + \xi^{(\beta)}\eta_{(\beta)}(a_{(\alpha)})^2 \tag{1.11}$$

Hence it follows that conditions (1.9) will be satisfied if

$$\xi^{(\alpha)} \neq -(a_{(\alpha)})^{-2}, \quad \alpha = 1, \ldots, N \tag{1.12}$$

Hence, condition (1.2) also follows from (1.8) and (1.10).

Thus, the compatibility conditions (1.9) and (1.10), the compatibility conditions (1.2) and the compatibility conditions (1.1) are equivalent to one another when inequalities (1.12) are satisfied.

2. Suppose R_{ij} is a positive-definite operator. It then follows from the condition

$$A_i \equiv R_{ij}(\sigma_{jk,k} + \rho F_j) = 0 \tag{2.1}$$

that

$$\sigma_{ij,j} + \rho F_i = 0 \tag{2.2}$$

We will form linear invariants of the tensor $A_{ij} \equiv A_{i,j} + A_{j,i}$

$$A_{(\alpha)} \equiv A_{ij}a_{ij}^{(\alpha)}, \quad \alpha = 1, \ldots, N \tag{2.3}$$

By analogy with (1.7) we have

$$A^\circ \equiv A_{ij}\delta_{ij} = \sum_{\alpha=1}^{N} A_{(\alpha)} \tag{2.4}$$

Obviously, to satisfy the conditions

$$A_{ij} = 0 \tag{2.5}$$

it is necessary and sufficient to satisfy the conditions

$$\overline{A}_{ij} \equiv A_{ij} + \sum_{\alpha=1}^{N} \xi^{(\alpha)} A_{(\alpha)} a_{ij}^{(\alpha)} = 0 \tag{2.6}$$

if inequalities (1.12) are satisfied.

Indeed, relation (2.6) follows from (2.3) and (2.4). Suppose now that we specify (2.6). We will convolute it with each tensor basis $a_{ij}^{(\beta)}$ ($\beta = 1, \ldots, N$). We then obtain from definitions (2.3) and (1.4)

$$\overline{A}_{ij} a_{ij}^{(\beta)} \equiv A_\beta + \xi^{(\beta)} A_{(\beta)} (a_{(\beta)})^2 = 0 \tag{2.7}$$

When condition (1.12) is satisfied it follows from (2.7) that

$$A_{(\alpha)} = 0, \ \alpha = 1, \ldots, N \tag{2.8}$$

Hence (2.5) follows from (2.6).

Suppose now that the defining relations of the mechanics of a strained solid (physically linear or non-linear) enable us to express the strain tensor ε_{ij} in terms of the stresses σ_{ij}. After making this replacement we will conventionally denote the tensor H_{ij} (1.8) in terms of $H_{ij}(\sigma)$.

We will form the tensor

$$\overline{H}_{ij}(\sigma) \equiv H_{ij}(\sigma) + \overline{A}_{ij} \tag{2.9}$$

It follows from (2.1) and (2.2) that the tensor H_{ij} satisfies the equations

$$\tag{2.10}$$

The new formulation of the problem of the mechanics of a strained solid for anisotropic media consists in finding a sufficiently smooth field of the stresses σ_{ij} which satisfy Eqs (2.10) in a simply-connected region of three-dimensional Euclidean space, when the following boundary conditions are satisfied on the boundary of this region—a certain smooth surface Σ

$$\sigma_{ij} n_j \big|_\Sigma = S_i^0 \ \ (\sigma_{ij,j} + \rho F_i)\big|_\Sigma = 0 \tag{2.11}$$

where n_j are the components of the unit vector of the outward normal to the surface F_i are the mass forces, and S_i^0 are the surface forces.

We will prove that the solution of problem (2.10) and (2.11) satisfies the equations of equilibrium (2.2) over the whole region as well as the conditions of compatibility (1.1).

To do this we will convolute expression (2.8) with each tensor of the basis $a_{ij}^{(\beta)}$, $\beta = 1, \ldots, N$. We then obtain from (1.11) and (2.7)

$$\overline{H}_{ij}(\sigma) a_{ij}^{(\beta)} \equiv (\eta_{(\beta)} + A_{(\beta)})(1 + \xi^{(\beta)} (a_{(\beta)})^2) = 0$$

whence, when condition (1.9) are satisfied, it follows that

$$\eta_{(\beta)} + A_{(\beta)} = 0$$

We will now differentiate (2.9) with respect to the jth coordinate

$$\overline{H}_{ij,j}(\sigma) = H_{ij,j}(\sigma) + \overline{A}_{ij,j} \tag{2.12}$$

Consider each of the terms on the right-hand side of (2.12). It follows from (1.3) and (1.8) that

$$H_{ij,j} = \tfrac{1}{2}\eta^{\circ}_{,i} + \sum_{\alpha=1}^{N} \xi^{(\alpha)} a_{ij}^{(\alpha)} \eta_{(\alpha),j}$$

and, taking (1.7) into account, we obtain

$$H_{ij,j} = \sum_{\alpha=1}^{N} (\tfrac{1}{2}\delta_{ij} + \xi^{(\alpha)} a_{ij}^{(\alpha)}) \eta_{(\alpha),j} \tag{2.13}$$

For the second term in (2.12) we have from (2.6), taking (2.4) into account

$$\bar{A}_{ij,j} = \Delta A_i + \sum_{\alpha=1}^{N} (\tfrac{1}{2}\delta_{ij} + \xi^{(\alpha)} a_{ij}^{(\alpha)}) A_{(\alpha),j} \tag{2.14}$$

Now substituting (2.14) and (2.13) into (2.12) we obtain

$$\bar{H}_{ij,j}(\sigma) = \Delta A_i + \sum_{\alpha=1}^{N} (\tfrac{1}{2}\delta_{ij} + \xi^{(\alpha)} a_{ij}^{(\alpha)}) (\eta_{(\alpha)} + A_{(\alpha)})_{,j} = 0 \tag{2.15}$$

However, according to (2.12) the sum on the right-hand side of (2.15) vanishes, and hence $\Delta A_i = 0$, i.e. A_i is a harmonic vector. On the boundary of the simply connected region considered this vector vanishes by (2.1) and (2.11). Consequently, it is also equal to zero inside this region. Hence, the equations of equilibrium (2.2) are satisfied everywhere in the region. Equation (1.10) then follows from (2.10) by (2.9), and from it we obtain (1.1) and (1.2).

3. The result obtained in Section 2, holds for any physically non-linear medium possessing anisotropy of the mechanical properties.

As an example consider a transversely isotropic medium. For such a medium the tensor basis consists of two tensors [6]

$$a_{ij}^{(1)} \equiv a_{ij} = \delta_{ij} - l_i l_j, \quad a_{(1)} = \sqrt{2}$$

$$a_{ij}^{(2)} \equiv l_i l_j, \quad a_{(2)} = 1$$

where the unit vector l_i represents the direction of the axis of transverse isotropy.

The strain tensor ε_{ij} can be represented in the form of the sum of four pairwise-orthogonal tensors

$$\varepsilon_{ij} = \tfrac{1}{2}\tilde{\theta} a_{ij} + \varepsilon^{\circ} l_i l_j + p_{ij} + 2q_{ij} \tag{3.1}$$

where the linear invariants of the strain tensor $\tilde{\theta}$ and ε^0 are formed by convoluting the strain tensor with the tensors of the basis

$$\tilde{\theta} \equiv a_{ij}\varepsilon_{ij}, \quad \varepsilon^{\circ} \equiv l_i l_j \varepsilon_{ij}$$

while the deviators p_{ij} and q_{ij} have the form

$$q_{ij} = \tfrac{1}{2}(\varepsilon_{ik} l_k l_j + \varepsilon_{jk} l_k l_i) - \varepsilon^{\circ} l_i l_j \tag{3.2}$$

$$p_{ij} = \tfrac{1}{2}(\varepsilon_{ik} a_{kj} + \varepsilon_{jk} a_{ki}) - \tfrac{1}{2}\tilde{\theta} a_{ij} - q_{ij}$$

Equations (2.2) follow from the identify

$$\varepsilon_{ij} = \tfrac{1}{2}(\varepsilon_{ik} a_{kj} + \varepsilon_{jk} a_{ki}) + \tfrac{1}{2} l_k (\varepsilon_{ik} l_j + \varepsilon_{jk} l_i)$$

Note also the useful identities

$$p_{ij}l_j = 0, \quad q_{ij}l_j = l_k q_{kj} a_{ji} = \tfrac{1}{2} l_k \varepsilon_{kj} a_{ji} \tag{3.3}$$

In the linear theory of elasticity the strains are related to the stresses by Hooke's law [7]

$$\varepsilon_{ij} = J_{ijkl}\sigma_{kl} \tag{3.4}$$

where J_{ijkl} is the elastic compliance tensor, which for a transversely anisotropic medium has five independent constants

$$J_{ijkl} = \mu_1 a_{ij}a_{kl} + \mu_2(a_{ij}l_kl_l + a_{kl}l_il_j) + \mu_3 l_il_jl_kl_l + \mu_4(a_{ik}a_{jl} + a_{il}a_{jk}) + \\ + \mu_5(a_{ik}l_jl_l + a_{il}l_jl_k + a_{jk}l_il_l + a_{jl}l_il_k) \tag{3.5}$$

Just like the strain tensor (3.1) we can represent the stress tensor in the form of the sum of four pairwise-orthogonal tensors

$$\sigma_{ij} = \tilde{\sigma}a_{ij} + \sigma^\circ l_il_j + P_{ij} + 2Q_{ij} \tag{3.6}$$

$$\tilde{\sigma} \equiv \tfrac{1}{2} a_{ij}\sigma_{ij}, \quad \sigma^\circ = l_il_j\sigma_{ij}$$

$$Q_{ij} = \tfrac{1}{2}(\sigma_{ik}l_kl_j + \sigma_{jk}l_kl_i) - \sigma^\circ l_il_j$$

$$P_{ij} = \tfrac{1}{2}(\sigma_{ik}a_{kj} + \sigma_{jk}a_{ki}) - \tilde{\sigma}a_{ij} - Q_{ij}$$

Now substituting expansions (3.1), (3.5) and (3.6) into (3.4) and carrying out the necessary convolutions, taking (1.5) and (3.3) into account, we obtain

$$\tfrac{1}{2}\tilde{\theta}a_{ij} + \varepsilon^\circ l_il_j + p_{ij} + 2q_{ij} = (\mu_1 + \mu_4)a_{ij}\tilde{\sigma} + \mu_2 l_il_j\tilde{\sigma} + \\ + (\mu_2 a_{ij} + \mu_3 l_il_j)\sigma^\circ + 2\mu_4 P_{ij} + 4\mu_5(Q_{im}l_ml_j + Q_{jm}l_ml_i) \tag{3.7}$$

Convoluting the left- and right-hand sides of (3.7) first with a_{ij} and then with l_il_j we obtain, respectively,

$$\tilde{\theta} = (\mu_1 + \mu_4)\tilde{\sigma} + \mu_2\sigma^\circ, \quad \varepsilon^\circ = \mu_2\tilde{\sigma} + \mu_3\sigma^\circ \tag{3.8}$$

Now subtracting the first equation of (3.8), multiplied by $1/2a_{ij}$, and the second equation of (3.8), multiplied by l_il_j, from (3.7), we obtain

$$p_{ij} + 2q_{ij} = 2\mu_4 P_{ij} + 4\mu_5(Q_{im}l_ml_j + Q_{jm}l_ml_i) \tag{3.9}$$

Convoluting the left- and right-hand sides of (3.9) with $a_{ik}a_{jl}$ and using (3.3), we obtain

$$p_{kl} = 2\mu_4 P_{kl} \tag{3.10}$$

Bearing in mind the identity

$$Q_{im}l_ml_j + Q_{jm}l_ml_i = Q_{ij}$$

we obtain form (3.9)

$$q_{ij} = 2\mu_5 Q_{ij} \tag{3.11}$$

Thus, for a transversely isotropic medium Eq. (3.4) is equivalent to the two relations (3.8), connecting the linear invariants of the strain and stress tensors, and to the two relations (3.10) and (3.11), which indicate the proportionality between the two deviators of the strain and stress tensors.

We will write Eqs (2.10) for a transversely isotropic medium. To do this we will represent the Laplace operator in the form

$$\Delta \equiv \delta_{ij}\partial_i\partial_j = (a_{ij} + l_i l_j)\partial_i\partial_j = \tilde{\Delta} + \Delta^\circ$$

$$\tilde{\Delta} \equiv a_{ij}\partial_i\partial_j, \quad \Delta^\circ \equiv (l_k\partial_k)^2$$

We have from (1.2)

$$\eta_{ij} = (\tilde{\Delta} + \Delta^\circ)(\tfrac{1}{2}\tilde{\theta}a_{ij} + \varepsilon^\circ l_i l_j + p_{ij} + 2q_{ij}) + \tilde{\theta}_{,ij} + \varepsilon^\circ{}_{,ij} - \tfrac{1}{2}\tilde{\theta}_{,kj}a_{ik} - \tag{3.12}$$

$$-\varepsilon^\circ_{,kj}l_i l_k - p_{ik,kj} - 2q_{ik,kj} - \tfrac{1}{2}\tilde{\theta}_{,kj}a_{jk} - \varepsilon^\circ_{,ki}l_j l_k - p_{jk,ki} - 2q_{jk,ki}$$

Then, by (1.6) we have

$$\eta_{(1)} \equiv \eta_{ij}a_{ij} = \tilde{\Delta}\varepsilon^\circ + \Delta^\circ\tilde{\theta} - 2q_{mn,mn} + \tilde{\Delta}\tilde{\theta} - 2p_{mn,mn} \tag{3.13}$$

$$\eta_{(2)} \equiv \eta_{ij}l_i l_j = \tilde{\Delta}\varepsilon^\circ + \Delta^\circ\tilde{\theta} - 2q_{mn,mn}$$

For simplicity we will use the operator R_{ij} (2.1) in the form

$$R_{ij} = a_{ij}R_{(1)} + l_i l_j R_{(2)}$$

Then, by (2.3), we have

$$A_{(1)} \equiv A_{ij}a_{ij} = 2R_{(1)}a_{ij}(\sigma_{ik,kj} + \rho F_{i,j}) \tag{3.14}$$

$$A_{(2)} \equiv A_{ij}l_i l_j = 2R_{(2)}l_i l_j(\sigma_{ik,kj} + \rho F_{i,j})$$

In (3.12) and (3.13) we express the strains in terms of the stresses using (3.8), (3.10) and substitute the results into (1.8) and (2.9).

Then, assuming

$$b_{ij} \equiv \tfrac{1}{2}(\mu_1 + \mu_4)\ a_{ij} + \mu_2 l_i l_j, \quad c_{ij} \equiv \tfrac{1}{2}\mu_2 a_{ij} + \mu_3 l_i l_j$$

we obtain Eqs (2.10) in the form

$$\Delta(b_{ij}\tilde{\sigma} + c_{ij}\sigma^\circ + 2\mu_4 P_{ij} + 4\mu_5 Q_{ij}) + (\mu_1 + \mu_2 + \mu_4)\tilde{\sigma}_{,ij} + (\mu_2 + \mu_3)\sigma^\circ_{,ij} - \tag{3.15}$$

$$-\ (b_{ik}\tilde{\sigma}_{,kj} + b_{jk}\tilde{\sigma}_{,ki}) - (c_{ik}\sigma^\circ{}_{,kj} + c_{jk}\sigma^\circ{}_{,ki}) - 2\mu_4(P_{ik,kj} + P_{jk,ki}) - 4\mu_5(Q_{ik,kj} + Q_{jk,ki}) +$$

$$+\ a_{ij}\xi^{(1)}[\mu_2\tilde{\Delta}\tilde{\sigma} + \mu_3\tilde{\Delta}\sigma^\circ + (\mu_1 + \mu_4)\ \Delta\tilde{\sigma} + \mu_2\Delta\sigma^\circ - 4\mu_4 P_{mn,mn} - 4\mu_5 Q_{mn,mn}] +$$

$$+\ (R_{(1)}a_{ik} + R_{(2)}l_i l_k)(\sigma_{kl,lj} + \rho F_{k,j}) + (R_{(1)}a_{jk} + R_{(2)}l_j l_k)(\sigma_{kl,li} + \rho F_{k,i}) +$$

$$+\, a_{ij}\xi^{(1)}A_{(1)} + l_i l_j\xi^{(2)}A_{(2)} = 0$$

(the quantities ($A_{(1)}$ and $A_{(2)}$ are defined in (3.14)).

Equations (3.15) contain four arbitrary constants, two of which, $\xi^{(1)}$ and $\xi^{(2)}$, are dimensionless, for which, by (1.12)

$$\xi^{(1)} \neq -½, \ \xi^{(2)} \neq -1$$

where $R_{(1)}$ and $R_{(2)}$ are non-zero and have the dimensions of elastic compliance.

4. For a physically non-linear medium Eqs (2.10) will be non-linear due to the non-linearity of the defining relations, i.e. the non-linearity of the first term in (2.9).

We will consider one of the possible forms of such defining relations using the example of a transversely isotropic medium.

We will first assume that the defining relations are potential, i.e. a scalar function W of the invariants of the stress tensor exists such that

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial \sigma_{ij}} + \frac{\partial W}{\partial \sigma_{ji}}\right) \tag{4.1}$$

Every symmetrical second-rank tensor for a transversely isotropic medium has five independent invariants [6]. Two of these are linear ($\tilde{\theta}$ and σ°), and two are "quadratic", which can be related to the intensities of the deviators

$$P \equiv \sqrt{P_{ij}P_{ij}}, \ Q \equiv \sqrt{2Q_{ij}Q_{ij}} \tag{4.2}$$

We will choose as the fifth invariant the determinant of the sum of the deviators P_{ij} and Q_{ij}, which we will denote by R

$$R = P_{ij}Q_{jk}Q_{ki} \tag{4.3}$$

Thus, we know a scalar function of the invariants

$$W = W(\tilde{\sigma}, \ \sigma^\circ, P, Q, R) \tag{4.4}$$

The derivatives of the invariants with respect to the stress tensor have the form

$$\frac{\partial \tilde{\sigma}}{\partial \sigma_{ij}} = \frac{1}{2}a_{ij}, \ \frac{\partial \sigma^\circ}{\partial \sigma_{ij}} = l_i l_j, \ \frac{\partial P}{\partial \sigma_{ij}} = \frac{P_{ij}}{P}, \ \frac{\partial Q}{\partial \sigma_{ij}} = \frac{Q_{ij}}{Q}, \tag{4.5}$$

$$\frac{\partial R}{\partial \sigma_{ij}} = \frac{1}{2}(P_{ik}Q_{kj} + P_{jk}Q_{ki}) + Q_{ik}Q_{kj} - \frac{1}{4}Q^2(l_i l_j + \frac{1}{2}a_{ij})$$

Using (4.5) we obtain from (4.1)

$$\varepsilon_{ij} = \frac{1}{2}\frac{\partial W}{\partial \tilde{\sigma}}a_{ij} + \frac{\partial W}{\partial \sigma^\circ}l_i l_j + \frac{\partial W}{\partial P}\frac{P_{ij}}{P} + \frac{\partial W}{\partial Q_{ij}}\frac{Q_{ij}}{Q} + \frac{\partial W}{\partial R}\frac{\partial R}{\partial \sigma_{ij}} \tag{4.6}$$

We act with the defining relations (4.6) in the same way as in Section 3 in the linear case. We convolute relations (4.6) successively with a_{ij} and then with $l_i l_j$. We have

$$\tilde{\theta} = \partial W / \partial \tilde{\sigma}, \ \varepsilon^\circ = \partial W / \partial \sigma^\circ \tag{4.7}$$

We then obtain from (4.6), (4.7) and (3.2)

$$p_{ij} + 2q_{ij} = \frac{\partial W}{\partial P}\frac{P_{ij}}{P} + \frac{\partial W}{\partial Q}\frac{Q_{ij}}{Q} + \frac{\partial W}{\partial R}\frac{\partial R}{\partial \sigma_{ij}} \tag{4.8}$$

Convoluting (4.8) with the tensor $a_{ik}a_{jl}$ and using (4.5) we obtain

$$p_{ij} = \frac{\partial W}{\partial P}\frac{P_{ij}}{P} + \frac{\partial W}{\partial R}\left[Q_{ik}Q_{kj} - \frac{1}{4}Q^2\left(l_i l_j + \frac{1}{2}a\right)\right] \tag{4.9}$$

Comparing (4.8) and (4.9) we obtain

$$q_{ij} = \frac{\partial W}{\partial Q}\frac{Q_{ij}}{Q} + \frac{1}{2}\frac{\partial W}{\partial R}(P_{ik}Q_{kj} + P_{jk}Q_{ki}) \tag{4.10}$$

Thus, the potential defining relations for a transversely isotropic medium (4.1) and (4.4) are equivalent to the defining relations (4.7), (4.9) and (4.10).

If the defining relations are quasilinear [6], W in (4.4) will be independent of the fifth invariant of (4.3). In this case relations (4.9) and (4.10) have the following respective form

$$p_{ij} = \frac{\partial W}{\partial P}\frac{P_{ij}}{P}, \quad q_{ij} = \frac{\partial W}{\partial Q}\frac{Q_{ij}}{Q} \tag{4.11}$$

It follows from (4.11) that

$$p = \frac{\partial W}{\partial P}, \quad q = \frac{\partial W}{\partial Q}$$

Hence, Eqs (4.11) can be written in the form

$$p_{ij} = \frac{p}{P}P_{ij}, \quad q_{ij} = \frac{q}{Q}Q_{ij} \tag{4.12}$$

If the defining relations are not potential, they can be written for the strain tensor (3.1) in a form which generalizes (4.9) and (4.10)

$$p_{ij} = f_1 P_{ij} / P + f_2 [Q_{ik}Q_{kj} - 1/4Q^2(l_i l_j + 1/2a_{ij})] \tag{4.13}$$

$$q_{ij} = f_3 Q_{ij} / Q + f_4(P_{ik}Q_{kj} + P_{jk}Q_{ki})$$

where the functions f_1, f_2, f_3 and f_4, like the linear invariations of the strain tensor ($\tilde{\theta}$ and ε°), depend on five invariants of the stress tensor $\tilde{\sigma}$, σ°, P, Q, R (3.4), (4.2) and (4.3).

If the defining relations are quasilinear, we must put $f_2 = f_4 = 0$ in (4.13), and they can be written in the form (4.12).

Note that the second term in (2.9) may also turn out to be non-linear due to the choice of the non-linear operator R_{ij} in (2.1). In this case the mass forces will occur non-linearly in (2.10), unlike Eqs (3.15).

If the defining relations depend on the temperature T, and the Duhamel–Neumann hypothesis holds, the temperature will occur linearly in (3.9) even in the case of physical non-linearity. The tensor η_{ij} (1.2) will be supplemented by the term η_{ij}^T, which for the transversely isotropic case is expressed as follows:

$$\eta_{ij.}^T = (\alpha_1 a_{ij} + \alpha_2 l_i l_j)\Delta T + (\alpha_1 + 2\alpha_2)\ T_{,ij} - \alpha_1(a_{ik}T_{,kj} + a_{jk}T_{,ki}) - \alpha_2 l_k(l_i T_{,kj} + l_j T_{,ki})$$

$$\eta^T_{(1)} = 2\alpha_1\Delta^\circ T + (2\alpha_1 + \alpha_2)\tilde{\Delta}T$$

$$\eta^T_{(2)} = 2\alpha_1\Delta^\circ T + \alpha_2\tilde{\Delta}T$$

Here the tensor of the thermal expansion α_{ij} can be represented in the form

$$\alpha_{ij} = \alpha_1 a_{ij} + \alpha_2 l_i l_j$$

In conclusion we note that for a transversely isotropic medium the differentiation operator ∂_i can also be represented in the form

$$\partial_i = \partial_i + l_i\partial^0; \quad \partial^0 = l_i\partial_i; \quad \tilde{\partial}_i = a_{ij}\partial_j$$

Hence we have

$$\Delta \equiv \delta_{ij}\partial_i\partial_j = (a_{ij} + l_i l_j)\partial_i\partial_j = \tilde{\partial}_i\tilde{\partial}_i + (\partial^\circ)^2 = \tilde{\Delta} + \Delta^\circ$$

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